

BAND FUNCTIONS IN THE PRESENCE OF MAGNETIC STEPS

P. D. HISLOP, N. POPOFF, N. RAYMOND, AND M. P. SUNDQVIST

ABSTRACT. We complete the analysis of the band functions for two-dimensional magnetic Schrödinger operators with piecewise constant magnetic fields. The discontinuity of the magnetic field can create edge currents that flow along the discontinuity that have been described by physicists. Properties of these edge currents are directly related to the behavior of the band functions. The effective potential of the fiber operator is an asymmetric double well (eventually degenerated) and the analysis of the splitting of the bands incorporates the asymmetry. If the magnetic field vanishes, the reduced operator has essential spectrum and we provide an explicit description of the band functions located below the essential spectrum. For non degenerate magnetic steps, we provide an asymptotic expansion of the band functions at infinity. We prove that when the ratio of the two magnetic fields is rational, a splitting of the band functions occurs and has a natural order, predicted by numerical computations.

AMS 2000 Mathematics Subject Classification: 35J10, 81Q10, 35P20

Keywords: magnetic Schrödinger operators, edge currents, band functions

1. INTRODUCTION AND MOTIVATION

In this paper, we investigate the spectral properties of magnetic Hamiltonians

$$\mathcal{L}_{\mathbf{b}} = (-i\nabla - \mathbf{A})^2 = D_x^2 + (D_y - a_{\mathbf{b}}(x))^2, \quad (1.1)$$

in $L^2(\mathbb{R}^2)$ with the following discontinuous magnetic field

$$\mathbf{B}(x, y) = b_1 \mathbf{1}_{\mathbb{R}_-}(x) + b_2 \mathbf{1}_{\mathbb{R}_+}(x),$$

where $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, and $\mathbf{1}_X(x)$ is the characteristic function on $X \subset \mathbb{R}$. An associated vector potential is given by :

$$\mathbf{A}(x, y) = (0, a_{\mathbf{b}}(x)), \quad a_{\mathbf{b}}(x) = b_1 x \mathbf{1}_{\mathbb{R}_-}(x) + b_2 x \mathbf{1}_{\mathbb{R}_+}(x).$$

These operators naturally appear in several models involved in nanophysics such as the Ginzburg–Landau model of superconductivity (see [31]) and in quantum transport in a bidimensional electron gas. In particular in an unbounded system, inhomogenous magnetic fields may play the role of a quantum waveguide ([23, 30]). More recently it appears that such magnetic barriers also induce transport and act as waveguides in graphene by creating snake states along magnetic discontinuities (see [22] and [11] where piecewise constant magnetic fields are considered). These physical properties can be described with the corresponding magnetic Schrödinger operators that exhibit a variety of interesting spectral and transport properties. In particular, there may exist edge currents that propagate along the edge at $x = 0$.

PDH was partially supported by NSF through grant DMS-1103104.
Version of October 31, 2016.

The conductivity corresponding to these edge currents is, in fact, quantized. Most of these properties can be deduced from the study of the band functions associated with the Hamiltonian (see for instance [21, 8]). We will describe later some consequences of our analysis of band functions.

The symmetries of the Hamiltonian allow us to classify various configurations of the magnetic field (b_1, b_2) . We notice that $\overline{\mathcal{L}_{\mathbf{b}}} = \mathcal{L}_{-\mathbf{b}}$ so that we may assume that $b_2 \geq 0$. If S denotes the symmetry $(x, y) \mapsto (-x, -y)$, we have $S\mathcal{L}_{b_1, b_2}S = \mathcal{L}_{-b_2, -b_1} = \overline{\mathcal{L}_{b_2, b_1}}$. For $B > 0$, we introduce the L^2 -unitary transform

$$U_B\psi(x, y) = B^{-1/2}\psi(B^{-1/2}x, B^{-1/2}y)$$

and we have

$$U_B^{-1}\mathcal{L}_{\mathbf{b}}U_B = B^{-1}\mathcal{L}_{B\mathbf{b}}.$$

These considerations allow the following reductions:

- (1) If b_1 or b_2 is zero we may assume that $b_1 = 0$ and, if $b_2 \neq 0$, we may assume that $b_2 = 1$. We call the case $(b_1, b_2) = (0, 1)$ the “magnetic wall”.
- (2) If b_1 and b_2 have opposite signs and $|b_1| \neq |b_2|$, it is sufficient to consider the case when $-1 < b_1 < 0 < b_2 = 1$. We call this case the “trapping magnetic step” (the term trapping will appear clearer after stating our results).
- (3) If $|b_1| = |b_2|$, we may restrict ourselves to the cases $(b_1, b_2) = (1, 1)$ and $(b_1, b_2) = (-1, 1)$. This last one may be called the “symmetric trapping magnetic step” or “magnetic barrier”.
- (4) If b_1 and b_2 have the same sign, we may assume that $0 < b_1 < b_2 = 1$. We call this case the “non-trapping magnetic step”.

A physical description of the distinct classical orbits associated with these magnetic field configurations is provided in [24, 30]. The last two cases, (3) and (4), have already been studied in the mathematical literature (see the overview below). Our goal is to provide a mathematical treatment of the unstudied cases (1) and (2). We will also give the consequences of our results on the behavior of quantum particles submitted to the magnetic field \mathbf{b} .

1.1. The band functions and their link with the magnetic Laplacian.

• *Direct fiber decomposition.* In order to perform the spectral analysis, we can use the translation invariance in the y -direction and thus the direct integral decomposition (see [29, XIII.16]) associated with the Fourier transform with respect to y , denoted by \mathcal{F}_y ,

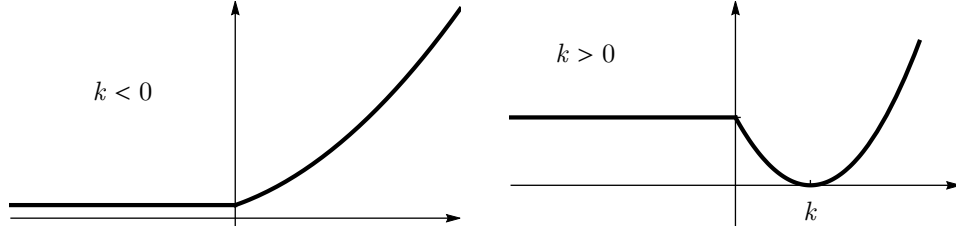
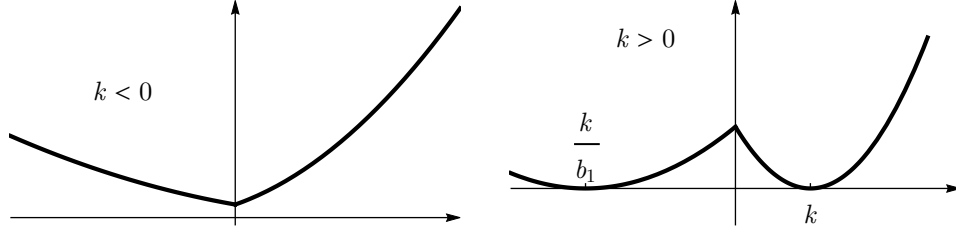
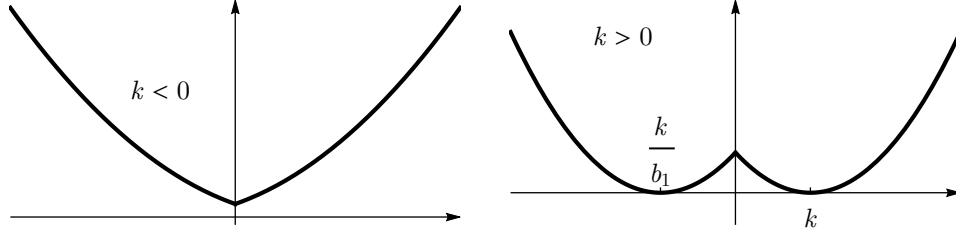
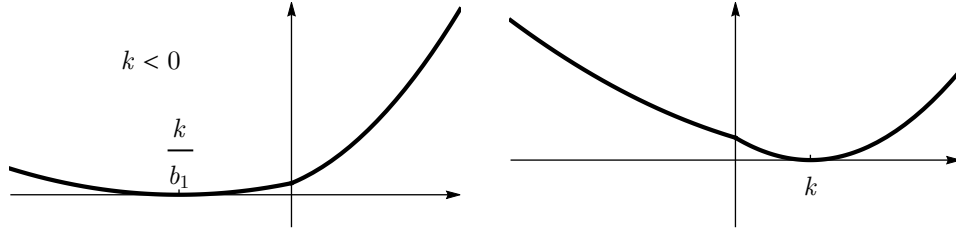
$$\mathcal{F}_y\mathcal{L}_{\mathbf{b}}\mathcal{F}_y^{-1} = \int_{k \in \mathbb{R}}^{\oplus} \mathfrak{h}_{\mathbf{b}}(k) dk,$$

where

$$\mathfrak{h}_{\mathbf{b}}(k) = D_x^2 + V_{\mathbf{b}}(x, k), \quad \text{with} \quad V_{\mathbf{b}}(x, k) = (k - a_{\mathbf{b}}(x))^2.$$

Note that the effective potential $V_{\mathbf{b}}(x, k)$ has the form

$$V_{\mathbf{b}}(x, k) = \begin{cases} (k - b_1x)^2, & x < 0, \\ (k - b_2x)^2, & x > 0. \end{cases} \quad (1.2)$$

FIGURE 1. The potential $V_{\mathbf{b}}(x, k)$ corresponding to case (1).FIGURE 2. The potential $V_{\mathbf{b}}(x, k)$ corresponding to case (2).FIGURE 3. The potential $V_{\mathbf{b}}(x, k)$ corresponding to case (3) with $b_1 = -1$ and $b_2 = 1$.FIGURE 4. The potential $V_{\mathbf{b}}(x, k)$ corresponding to case (4).

The domain of $\mathfrak{h}_{\mathbf{b}}(k)$ is given by

$$\text{Dom}(\mathfrak{h}_{\mathbf{b}}(k)) = \{\psi \in \text{Dom}(\mathfrak{q}_{\mathbf{b}}(k)) : (D_x^2 + V_{\mathbf{b}}(x, k))\psi \in \mathcal{L}^2(\mathbb{R})\},$$

where the quadratic form $\mathfrak{q}_{\mathbf{b}}(k)$ is defined by

$$\mathfrak{q}_{\mathbf{b}}(k)(\psi) = \int_{x \in \mathbb{R}} |\psi'(x)|^2 + V_{\mathbf{b}}(x, k)|\psi|^2 dx.$$

The spectrum of $\mathcal{L}_{\mathfrak{b}}$, $\text{sp}(\mathcal{L}_{\mathfrak{b}})$, satisfies

$$\text{sp}(\mathcal{L}_{\mathfrak{b}}) = \overline{\bigcup_{k \in \mathbb{R}} \text{sp}(\mathfrak{h}_{\mathfrak{b}}(k))}.$$

We now define the n -th band function below the essential spectrum:

Notation 1.1. We denote by $\lambda_{\mathfrak{b},n}(k)$ the n -th Rayleigh quotient of $\mathfrak{h}_{\mathfrak{b}}(k)$. We recall that if $\lambda_{\mathfrak{b},n}(k)$ is strictly less than the infimum of the essential spectrum, it coincides with the n -th eigenvalue of $\mathfrak{h}_{\mathfrak{b}}(k)$.

- *State of the art.* The cases (3) and (4) have already been extensively studied. Case (3) for which $(b_1, b_2) = (1, 1)$ is just the Landau Hamiltonian. Its spectrum is pure point and consists of the so-called Landau levels; infinitely degenerated eigenvalues $(2n - 1)$, for $n \geq 1$. The band functions are constants $\lambda_{\mathfrak{b},n}(k) = 2n - 1$.

The case (3), for which $(b_1, b_2) = (-1, 1)$, can be linked to operators acting on a half-line. We first need to introduce the standard de Gennes operators, that will also be used in some results of this article. We recall below well-known properties of the spectrum of these operators.

Notation 1.2. We let $E_0 = 0$ and for $n \geq 1$, $E_n = 2n - 1$.

Notation 1.3. For $\circ = \text{N}$, (resp. $\circ = \text{D}$), let us introduce $\mathfrak{h}^{\circ}(k)$ the Neumann (resp. Dirichlet) realization on \mathbb{R}_+ of $D_t^2 + (t - k)^2$ and its eigenvalues $(\mu_{\ell}^{\circ}(k))_{\ell \geq 1}$. For all $\ell \geq 1$, the function μ_{ℓ}^{N} admits a unique and non-degenerate minimum at $k = \xi_{\ell-1}$, denoted by $\Theta_{\ell-1}$, with $\Theta_{\ell-1} \in (E_{\ell-1}, E_{\ell})$. Moreover, $\xi_{\ell-1}$ is also the unique solution of the equation $\mu_{\ell}^{\text{N}}(k) = k^2$. The function $\mu_{\ell}^{\text{D}}(k)$ is a decreasing function of k . Both $\mu_{\ell}^{\text{N}}(k)$ and $\mu_{\ell}^{\text{D}}(k)$ go to $+\infty$ as $k \rightarrow -\infty$ and go to E_{ℓ} as $k \rightarrow +\infty$.

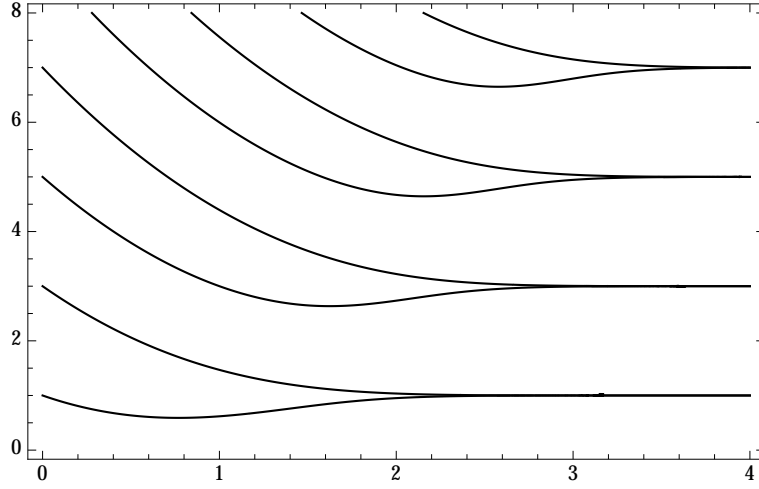
We refer mainly to [6] for the Dirichlet operator and [5, 13] for the Neumann one. All these properties can also be found in [10, 19].

Using symmetry arguments, it is proved in [9, 25] that for $\mathfrak{b} = (-1, 1)$ and $\ell \geq 1$,

$$\lambda_{\mathfrak{b},2\ell-1}(k) = \mu_{\ell}^{\text{N}}(k) \quad \text{and} \quad \lambda_{\mathfrak{b},2\ell}(k) = \mu_{\ell}^{\text{D}}(k) \quad k \in \mathbb{R},$$

linking the de Gennes operator with the model with symmetric trapping magnetic steps. This analysis of the band functions provides a description of a quantum particle localized in energy far from the thresholds $\{E_n\}$ and submitted to the system: they can exhibit edge current that may be viewed as quantum manifestations of the classical snake orbits [30] along the edge $x = 0$. These edge currents are exponentially decreasing far from the edge. The corresponding edge currents and the asymptotics of the band functions are described in [9]. The effective potential (see (1.2)) has the form of two symmetric double wells for $k > 0$. The fact that each band function is exponentially close to a Landau level as $k \rightarrow +\infty$ is called a *splitting* of the band function. The analysis of quantum states localized in energy near the thresholds is made in [16] for one edge quantum hall system and relies on a precise asymptotic analysis of the band function $\mu_{\ell}^{\text{D}}(k)$ as $k \rightarrow +\infty$. The band functions associated to this case are displayed in Figure 5.

The case (4), a non-trapping magnetic step, was treated in the initial paper by Iwatsuka [20] (along with the more general situation for which the magnetic field depends only on one direction and simply has constant values at $\pm\infty$). In this case, all the band functions are monotone nondecreasing. The analysis of the associated edge current is done in [17].

FIGURE 5. Band functions for the symmetric double-well case $\mathbf{b} = (-1, 1)$.

Notice that all cases (1)–(4) are described from a physical point of view in [30].

1.2. Main results. We first consider case (1) for which $(b_1, b_2) = (0, 1)$. This was not treated in [18] or in [9]. It is physically interesting because the classical orbits of an electron originating in the half-plane $x < 0$ are unbounded straight half-lines or lines (depending upon whether the orbit intersects $x = 0$ or not), whereas those in the half-plane $x > 0$ are bounded circles provided the Lamor orbit does not cross the edge at $x = 0$.

Theorem 1.4. *Let $\mathbf{b} = (0, 1)$.*

If $k < 0$, then the fiber operator $\mathfrak{h}_{\mathbf{b}}(k)$ has only purely absolutely continuous spectrum $[k^2, \infty)$. Consequently, the spectrum of $\mathcal{L}_{\mathbf{b}}$ is $[0, \infty)$.

If $k > 0$, the essential spectrum of $\mathfrak{h}_{\mathbf{b}}(k)$ is $[k^2, \infty)$. The band functions $\lambda_{\mathbf{b},n}$ are nondecreasing functions with $E_{n-1} < \lambda_{\mathbf{b},n}(k) < E_n$ and $\lim_{k \rightarrow \infty} \lambda_{\mathbf{b},n}(k) = E_n = 2n - 1$. For $n \geq 1$ and $\xi_{n-1} < k < \xi_n$ (recall Notation 1.3), the operator $\mathfrak{h}_{\mathbf{b}}(k)$ admits exactly n simple eigenvalues below the threshold of its essential spectrum.

The case (2) of a trapping magnetic step is technically more challenging. It relies on the asymptotics of the eigenfunctions of a Schrödinger operator with an asymmetric double-well potential. The results concerning the splitting of two energy levels depends upon the ratio of the two constant magnetic fields $b_1/b_2 = b_1$ since $b_2 = 1$ by normalization.

Definition 1.5. We define $\mathfrak{L} = \{E_n, n \geq 1\} \cup \{|b_1|E_n, n \geq 1\}$ and we define the splitting set $\mathfrak{S} = \{E_n, n \geq 1\} \cap \{|b_1|E_n, n \geq 1\}$. We will say that there is a *splitting of levels* if $\mathfrak{S} \neq \emptyset$ i.e. if there exist positive integers $m < n$ such that

$$b = \frac{2n-1}{2m-1}, \quad \text{with } b = |b_1|. \quad (1.3)$$

As we shall see, the problem of the behavior for large k of the band functions can be reformulated into a semiclassical problem for a one dimensional Schrödinger operator with multiple wells (see for instance [14, 15]). In particular the limits

of the band functions are precisely the elements of \mathfrak{L} , counted with multiplicity. Roughly speaking, either one or two band functions converge toward each element of \mathfrak{L} .

However our investigation will not rely on the semiclassical tools: We will use asymptotic expansions of special functions in order to derive the asymptotic behavior of the band functions. Let us now state our main theorem.

Theorem 1.6. *Let $\mathfrak{b} = (b_1, 1)$, with $-1 < b_1 < 0$. For $k \in \mathbb{R}$, the fiber operator $\mathfrak{h}_{\mathfrak{b}}(k)$ has only purely discrete spectrum $\lambda_{\mathfrak{b},n}(k)$ with simple eigenvalues, and $\lim_{k \rightarrow -\infty} \lambda_{\mathfrak{b},n}(k) = +\infty$. The set of limit points of the band functions $\lambda_{\mathfrak{b},n}(k)$ as $k \rightarrow +\infty$ is precisely \mathfrak{L} . Moreover, we may describe the asymptotics of the band functions in the following way.*

(i) *Non-splitting case. Let $\lambda \in \mathfrak{L} \setminus \mathfrak{S}$. We can write in a unique way $\lambda = E_n$ or $\lambda = bE_n$ with $n \geq 1$. Then there exists a unique $p_n \in \mathbb{N}^*$ such that:*

- (a) $\lambda_{\mathfrak{b},p_n}(k) - E_n = \varepsilon_n(k) \xrightarrow{k \rightarrow +\infty} 0$ (in the case $\lambda = E_n$),
(b) $\lambda_{\mathfrak{b},p_n}(k) - bE_n = \varepsilon_n(k) \xrightarrow{k \rightarrow +\infty} 0$ (in the case $\lambda = bE_n$).

In the first case,

$$\varepsilon_n(k) = -\frac{1}{\sqrt{\pi}} 2^{n-2} (1+b) \frac{1}{(n-1)!} k^{2n-3} e^{-k^2} (1 + o(1)),$$

and in the second case,

$$\varepsilon_n(k) = -\frac{1}{\sqrt{\pi}} 2^{n-2} b^{3/2-n} (1+b) \frac{1}{(n-1)!} k^{2n-3} e^{-k^2/b} (1 + o(1)).$$

(ii) *Splitting case. Assume that the splitting set is not empty and consider $\lambda \in \mathfrak{S}$. Let us consider $n \geq 1$ and $m \geq 1$ such that $\lambda = E_n = bE_m$. There exists $p_n \in \mathbb{N}^*$ such that*

$$\begin{cases} \lambda_{\mathfrak{b},p_n}(k) - E_n = \varepsilon_n^-(k) \\ \lambda_{\mathfrak{b},p_n+1}(k) - E_n = \varepsilon_n^+(k) \end{cases}$$

with, as k tends to $+\infty$,

$$\begin{aligned} \varepsilon_n^-(k) &= -\frac{2^{n-3/2} (1+b)}{(n-1)! \sqrt{\pi}} k^{2n-3} e^{-k^2} (1 + o(1)), \\ \varepsilon_n^+(k) &= \frac{2^{m+\frac{3}{2}} b^{-m+\frac{3}{2}}}{(m-1)! (1+b) \sqrt{\pi}} k^{2m+1} e^{-k^2/b} (1 + o(1)). \end{aligned}$$

The energy levels for which the exponentially small splitting occurs when $k \rightarrow \infty$ behave like the band functions in the symmetric magnetic field case case $b_1 = -1$ and $b_2 = 1$. In that case (see [9]), two band functions approach the Landau level $2n - 1$ as $k \rightarrow \infty$, one is monotone decreasing and the other has a unique, nondegenerate minimum (see Figure 5). In the present article, with the help of our asymptotic expansions, we may deduce the existence of a minimum for some band functions.

Corollary 1.7. *With the same assumptions and notations as in Theorem 1.6, the following statements hold:*

- (i) *Let $\lambda \in \mathfrak{L} \setminus \mathfrak{S}$. Then each band function converging to λ admits a global minimum.*
(ii) *Let $\lambda \in \mathfrak{S}$. Then $k \mapsto \lambda_{p_n}(k)$ admits a global minimum.*

Notice that the first band function always admits a global minimum. Moreover if $|b_1|$ is not a ratio of odd integers, all the band functions admit a global minimum. We conjecture that the only case when monotonicity occurs is Case (ii) of Theorem 1.6 and corresponds to λ_{p_n+1} . In that case, the non-monotonicity of each band function depends on a non trivial way on the index of the eigenvalue. This situation seems striking to us, especially compared to other known case (see [9, 17, 34]).

1.3. Numerical computations. We display on Figures 6–8 the band functions associated with the magnetic step $\mathbf{b} = (-\frac{1}{3}, 1)$ together with the asymptotics provided below as $k \rightarrow +\infty$. The splitting occurs near each value in $\mathfrak{S} = \{2n - 1, n \in \mathbb{N}^*\}$. These graphs have been obtained by using Mathematica.

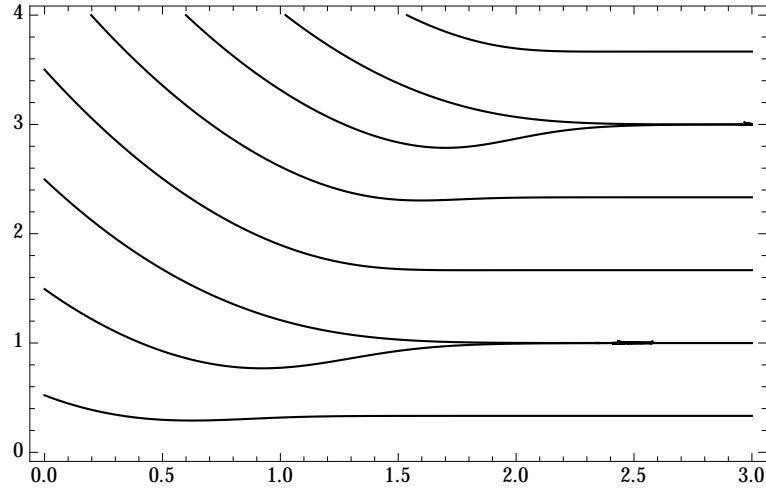


FIGURE 6. Band functions $k \mapsto \lambda_{\mathbf{b},n}(k)$ for $\mathbf{b} = (-\frac{1}{3}, 1)$.

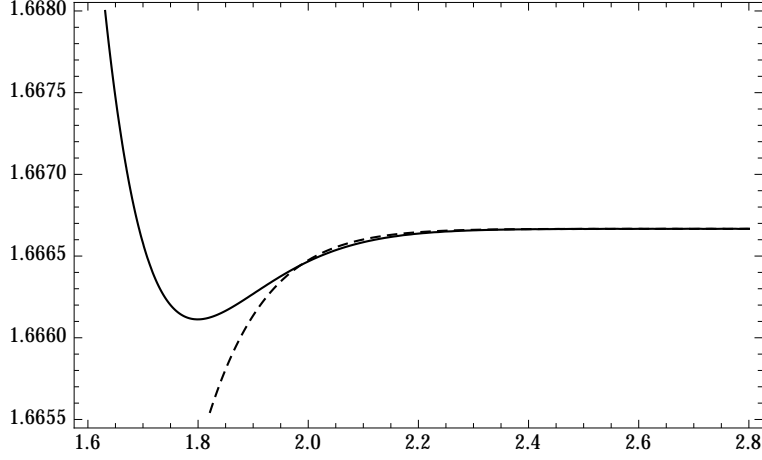


FIGURE 7. Zoom near $\lambda = \frac{5}{3}$ on the band function $k \mapsto \lambda_{\mathbf{b},4}(k)$ for $\mathbf{b} = (-\frac{1}{3}, 1)$. In dash line: the asymptotics from Theorem 1.6, case (ib). Here $n = 3$.

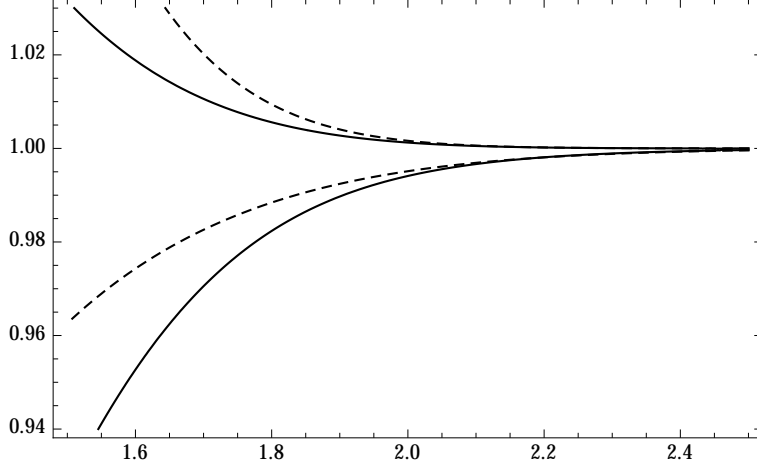


FIGURE 8. Zoom near $\lambda = 1$ on the band functions $k \mapsto \lambda_{\mathbf{b},p}(k)$ for $\mathbf{b} = (-\frac{1}{3}, 1)$ and $p = 2, 3$ (splitting case). In dash line: the asymptotics from Theorem 1.6, case (ii). Here $\lambda = 1$, $n = 1$ and $m = 2$.

1.4. Consequence of the analysis of the band functions. In this section we qualitatively present possible consequences of our results.

- *Standard perturbation of the Hamiltonian.* If we perturb the Hamiltonian $\mathcal{L}_{\mathbf{b}}$ by a suitable electric potential, discrete spectrum might appear under the essential spectrum, which is the infimum of the first band function. In particular, the existence of a minimum is useful if one wants to study the counting function of created eigenvalues. Here we know that the infimum is reached for some finite frequency

k^* . Then, following the standard procedure described for example in [27], the number of discrete eigenvalues can be described, depending on the degeneracy of the minimum and on the decay of the electric perturbation.

In the same way, it is possible to derive a limiting absorption principle at energies close to the bottom of the spectrum of \mathcal{L}_b (and also near each band minimum), as in [33]. The case of thresholds corresponding to the limits of band functions can also be studied in the spirit of [26]. In particular the asymptotics of the band functions for large k allow to describe the suitable functional spaces when writing a limiting absorption principle near the Landau levels.

- *Semiclassical magnetic Laplacian with piecewise constant magnetic field.* Let us consider the semiclassical magnetic Laplacian $(-i\hbar\nabla + \mathbf{A})^2$ in the plane with a magnetic field such that

$$\nabla \times \mathbf{A} = \mathbf{B} = b_1 \mathbb{1}_\Omega + b_2 \mathbb{1}_{\mathbb{C}\Omega},$$

where Ω is a smooth and bounded domain. It is well-known that the semiclassical analysis of the low lying spectrum (motivated by the analysis of the third critical field in the Ginzburg–Landau theory) is governed by the spectral behavior of some local models (see [10, 3]). When the magnetic field is not continuous, models operators in the form of \mathcal{L}_b have to be considered. It is known that the minima (which are often non degenerate) of the corresponding band functions may combine with the curvature (of the boundary for instance) to generate the semiclassical asymptotics. In particular, the existence of a minimum of the band functions implies a microlocalization of the magnetic eigenfunctions (see [10, 28, 4]).

- *Bulk states localized in energy near the thresholds.* In general Iwatsuka Hamiltonians (*i.e.* Hamiltonians associated with magnetic fields depending only on one variable), the analysis of quantum states submitted to the magnetic field and localized in energy in an interval I depends on the behavior of the band functions when crossing the energy interval I , see [21]. For example if I does not contain any threshold, then the quantum states are localized in space near the variations of the magnetic field (here, the jump line $\{x = 0\}$) and their current (defined below) is not small. This corresponds to the fact that a classical charged particle moves along the edge $\{x = 0\}$, in a “snake orbit”.

One may also show, as in [6, 16], that there exist states, called bulk states, localized in energy near the thresholds, bearing small current. If the threshold corresponds to a limit of a band function, these bulk states are exponentially small near the jump line. The precise (de)localization of these states and the current they bear can be derived from the asymptotics of the band function by the following method (see [16]). Assume that $I = (\lambda - \delta, \lambda + \delta)$ where $\lambda \in \mathfrak{S}$ and $\delta \downarrow 0$. The current of a state $\psi \in \mathbb{L}^2(\mathbb{R})$ along the jump line is given by $\langle J_y \psi, \psi \rangle$ where $J_y := \frac{i}{2}[\mathcal{L}_b, y] = -i\partial_y - a_b(x)$ is the current operator. This quantity may be linked to the derivative of the band functions by using the Feynman–Hellmann formula. Using the asymptotics from Theorem 1.6, one may show that the derivative of the band functions is small on the large k interval corresponding to the values where $\lambda_n(k) \in I$. This would provide an upper bound on the current carried by bulk states, depending on how far from the threshold is their energy. Then follows an analysis in phase space of these states in order to show that delocalization in the frequency k involves delocalization in the x -variable.

Note finally that the sign of the derivative of the band function is related to the direction of propagation of the particle along the jump line of the magnetic field. In particular quantum states related to non-monotonic band functions (see Corollary 1.7) may flow along the line in two possible directions. This corresponds to the possible *negative velocity* described in [30]. We refer to [34] for an analysis of the direction of propagation of quantum states in a translation invariant magnetic field.

2. MAGNETIC WALL

This section is devoted to the case $\mathbf{b} = (0, 1)$. Away from the edge at $x = 0$, a classical electron moves in a straight line for $x < 0$ and in a closed circular orbit for $x > 0$. If the orbit intersects the edge, then the classical electron escapes to $x = -\infty$. The absence of magnetic field for $x < 0$ results in a half-line of essential spectrum for the quantum electron. As we will see, the spectrum varies with the sign of k .

The reduced potential $V_{\mathbf{b}}$ satisfies $V_{\mathbf{b}}(x, k) = k^2$ for $x \leq 0$, and $V_{\mathbf{b}}(x, k) \rightarrow +\infty$ as $x \rightarrow +\infty$. As a consequence, we easily deduce the following proposition.

Proposition 2.1. *Assume that $\mathbf{b} = (0, 1)$ and $k \in \mathbb{R}$. The essential spectrum of $\mathfrak{h}_{\mathbf{b}}(k)$ is given by*

$$\mathrm{sp}_{\mathrm{ess}}(\mathfrak{h}_{\mathbf{b}}(k)) = [k^2, +\infty).$$

Moreover, $\mathrm{sp}(\mathcal{L}_{\mathbf{b}}) = [0, +\infty)$.

In fact, we can prove slightly more.

Proposition 2.2. *Assume that $\mathbf{b} = (0, 1)$ and $k \in \mathbb{R}$. The operator $\mathfrak{h}_{\mathbf{b}}(k)$ has no embedded eigenvalues in its essential spectrum.*

Proof. Let us consider $\lambda \geq k^2$ and $\psi \in \mathrm{Dom}(\mathfrak{h}_{\mathbf{b}}(k))$ such that:

$$-\psi'' + (k - a_{\mathbf{b}}(x))^2\psi = \lambda\psi. \quad (2.4)$$

For $x < 0$, we have $-\psi'' = (\lambda - k^2)\psi$ whose only solution in $L^2(\mathbb{R}_-)$ is zero. But since the solutions of (2.4) belongs to H_{loc}^2 and are in $\mathcal{C}^1(\mathbb{R})$, this implies that $\psi(0) = \psi'(0) = 0$ and thus $\psi \equiv 0$ by the Cauchy–Lipschitz theorem. \square

Let us now describe the discrete spectrum, that is the eigenvalues $\lambda < k^2$. Since, for $k \leq 0$, we have $\mathfrak{q}_{\mathbf{b}}(k) \geq k^2$, we deduce the following proposition by the min-max principle.

Proposition 2.3. *Assume that $\mathbf{b} = (0, 1)$ and $k \leq 0$. Then*

$$\mathrm{sp}(\mathfrak{h}_{\mathbf{b}}(k)) = \mathrm{sp}_{\mathrm{ess}}(\mathfrak{h}_{\mathbf{b}}(k)) = [k^2, +\infty).$$

Therefore we must only analyze the case when $k > 0$. The following lemma is a reformulation of the eigenvalue problem.

Lemma 2.4. *Assume that $\mathbf{b} = (0, 1)$ and $k > 0$. The number $\lambda < k^2$ is an eigenvalue of $\mathfrak{h}_{\mathbf{b}}(k)$ if and only if there exists a non zero function $\psi \in L^2(\mathbb{R}_+)$ satisfying*

$$\begin{cases} -\psi''(x) + (x - k)^2\psi = \lambda\psi(x), & (2.5a) \\ \psi'(0) - \sqrt{k^2 - \lambda}\psi(0) = 0. & (2.5b) \end{cases}$$

Moreover, the eigenfunctions of $\mathfrak{h}_{\mathbf{b}}(k)$ can only vanish on \mathbb{R}_+ . The eigenvalues of $\mathfrak{h}_{\mathbf{b}}(k)$ are simple.

Proof. We consider $\mathfrak{h}_{\mathbf{b}}(k)\psi = \lambda\psi$. Since $\psi \in L^2(\mathbb{R}_-)$, there exists $A \in \mathbb{R}$ such that, for $x \leq 0$, $\psi(x) = Ae^{x\sqrt{k^2-\lambda}}$. Then we have to solve $-\psi'' + V_{\mathbf{b}}(x, k)\psi = \lambda\psi$ for $x \geq 0$ with the transmission conditions

$$\psi(0) = A \quad \text{and} \quad \psi'(0) = A\sqrt{k^2 - \lambda}$$

or equivalently

$$\psi'(0) - \sqrt{k^2 - \lambda}\psi(0) = 0.$$

In particular, A cannot be zero. The simplicity is a consequence of the Cauchy–Lipschitz theorem. \square

Lemma 2.5. *Assume that $\mathbf{b} = (0, 1)$. The functions $\mathbb{R}_+ \ni k \mapsto \lambda_{\mathbf{b},n}(k)$ are non decreasing.*

Proof. We use the translation $x = y + k$ to see that $\mathfrak{h}_{\mathbf{b}}(k)$ is unitarily equivalent to $D_y^2 + \tilde{V}(y, k)$ with $\tilde{V}(y, k) = \mathbf{1}_{(-\infty, -k)}(y)k^2 + \mathbf{1}_{(-k, +\infty)}(y)y^2$. For $0 < k_1 < k_2$, we have:

$$\begin{aligned} \tilde{V}(y, k_2) - \tilde{V}(y, k_1) &= \mathbf{1}_{(-k_2, -k_1)}(y)y^2 + \mathbf{1}_{(-\infty, -k_2)}(y)k_2^2 - \mathbf{1}_{(-\infty, -k_1)}(y)k_1^2 \\ &\geq \mathbf{1}_{(-k_2, -k_1)}(y)k_1^2 + \mathbf{1}_{(-\infty, -k_2)}(y)k_2^2 - \mathbf{1}_{(-\infty, -k_1)}(y)k_1^2 \\ &= \mathbf{1}_{(-\infty, -k_2)}(y)k_2^2 - \mathbf{1}_{(-\infty, -k_2)}(y)k_1^2 \geq 0. \end{aligned}$$

By the min-max principle, we infer the desired monotonicity. \square

The next lemma is a consequence of the Sturm–Liouville theory (see for instance [32]).

Lemma 2.6. *Assume that $\mathbf{b} = (0, 1)$, and let $n \in \mathbb{N}$ be such that $\lambda_{\mathbf{b},n}(k) < k^2$. Then, the corresponding eigenspace is one dimensional and is generated by a normalized function $\psi_{\mathbf{b},n}(k)$, depending analytically on k . Moreover, $\psi_{\mathbf{b},n}(k)$ has exactly $n - 1$ zeros, all located on the positive half line.*

Proof. We have only to explain the part of the statement concerned with the zeros. Thanks to Lemma 2.4, one knows that the zeros are necessarily positive. Let us briefly recall the main lines of the Sturm’s oscillation theorem. Let us denote by $m - 1$ the number of zeros of $\psi = \psi_{\mathbf{b},n}(k)$ and let us check that $m \leq n$. For $j = 1, \dots, m - 1$, $z_{n,j}$ denotes the j -th zero of $\psi_{\mathbf{b},n}(k)$ (the zeros are simple and isolated by the Cauchy–Lipschitz theorem) and we let $\varphi_j = \psi|_{(z_{n,j}, z_{n,j+1})}$ extended by zero. By definition,

$$\mathfrak{q}_{\mathbf{b}}(k)(\varphi_j, \varphi_\ell) = \int_{\mathbb{R}} \varphi_j' \varphi_\ell' + V_{\mathbf{b}}(x, k) \varphi_j \varphi_\ell \, dx,$$

so,

$$\mathfrak{q}_{\mathbf{b}}(k)(\varphi_j, \varphi_\ell) = \delta_{j,\ell} \int_{\mathbb{R}} \varphi_j'^2 + V_{\mathbf{b}}(x, k) \varphi_j^2 \, dx = \lambda_{\mathbf{b},n} \delta_{j,\ell} \|\varphi_j\|^2.$$

We apply the min-max principle to the space $F_m = \text{span}_{1 \leq j \leq m} \varphi_j$ to get $\lambda_{\mathbf{b},m} \leq \lambda_{\mathbf{b},n}$ and thus $m \leq n$. In particular we recover that $\psi_{\mathbf{b},1}$ does not vanish. Let us explain why $\psi_{\mathbf{b},2}$ vanishes exactly once. Introducing the Wronskian $W_{1,2} = \psi_1 \psi_2' - \psi_2 \psi_1'$, we notice that $W_{1,2}' = (\lambda_{\mathbf{b},1} - \lambda_{\mathbf{b},2}) \psi_{\mathbf{b},1} \psi_{\mathbf{b},2}$. If $\psi_{\mathbf{b},2} > 0$, then $W_{1,2}' < 0$. Since the eigenfunctions belong to the Schwartz class at infinity, $W_{1,2}(x) \rightarrow 0$ as $|x| \rightarrow +\infty$. This is a contradiction. With the same kind of monotonicity arguments, we can prove that the zeros of the eigenfunctions are interlaced and that $\psi_{\mathbf{b},n}$ has exactly $n - 1$ zeros. \square

By using the harmonic approximation in the semiclassical limit (see [7] and the Section 3.1 below), we can prove the following lemma.

Lemma 2.7. *Assume that $\mathfrak{b} = (0, 1)$. For all $n \geq 1$, we have*

$$\lim_{k \rightarrow +\infty} \lambda_{\mathfrak{b},n}(k) = E_n.$$

In particular, for k large enough, we have $\lambda_{\mathfrak{b},n}(k) \leq E_n < k^2$.

Let us now prove that the n -th band function lies between the two consecutive Landau levels E_{n-1} and E_n .

Proposition 2.8. *Assume that $\mathfrak{b} = (0, 1)$. For all $n \geq 1$ and for all $k > 0$ such that $\lambda_{\mathfrak{b},n}(k) < k^2$ we have $\lambda_{\mathfrak{b},n}(k) \in (E_{n-1}, E_n)$.*

Proof. By Lemmas 2.5 and 2.7, $\lambda_{\mathfrak{b},n}(k) < E_n$ (the strict inequality comes from the analyticity). It remains to prove that $\lambda_{\mathfrak{b},n}(k) > E_{n-1}$. Clearly $\lambda_{\mathfrak{b},1}(k) > E_0$. We rely on Notation 1.3. For $n \geq 2$, we consider the function $\varphi_n(x) = \psi_{\mathfrak{b},n}(x + z_{n,1}(k))$ which satisfies $\mathfrak{h}^D(k - z_{n,1}(k))\varphi_n = \lambda_{\mathfrak{b},n}(k)\varphi_n$ and has exactly $n-2$ zeros on \mathbb{R}_+ , cf. Lemma 2.6. By the Sturm's oscillation theorem, φ_n is the $(n-1)$ -th eigenfunction of $\mathfrak{h}^D(k - z_{n,1}(k))$. Therefore we have $\lambda_{\mathfrak{b},n}(k) = \mu_{n-1}^D(k - z_{n,1}(k))$. Moreover for all $\ell \geq 1$ and $k \in \mathbb{R}$, $\mu_\ell^D(k) > E_\ell$. This follows by domain-monotonicity, once we notice that $\mathfrak{h}^D(k)$, after a translation, is given by the harmonic oscillator $D_x^2 + x^2$ on $(-k, +\infty)$ with Dirichlet boundary condition at $x = -k$, together with the fact that the harmonic oscillator $D_x^2 + x^2$ on \mathbb{R} has simple eigenvalues E_ℓ . This provides the desired conclusion. \square

We recall that the sequence $(\xi_n)_{n \geq 1}$ is defined in Notation 1.3. For the next proposition, we will need to use the so-called parabolic cylinder functions. We denote by $U(a, x)$ the first Weber parabolic cylinder function (whose properties are summarized in the Appendix, section A; for more details see, for example, [1, section 12] or [2]) which is a solution of the linear ordinary differential equation:

$$-y''(x) + \frac{1}{4}x^2y(x) = -ay(x). \quad (2.6)$$

The Weber function $U(a, x)$ decays exponentially when $x \rightarrow +\infty$.

Proposition 2.9. *Assume that $\mathfrak{b} = (0, 1)$. For all $n \geq 1$, the equation $\lambda_{\mathfrak{b},n}(k) = k^2$ has a unique non negative solution, $k = \xi_{n-1}$, such that, locally, for $k > k_n$, $\lambda_{\mathfrak{b},n}(k) < k^2$. Moreover we have $\lambda_{\mathfrak{b},n}(\xi_{n-1}) = \Theta_{n-1}$.*

Proof. Thanks to Lemma 2.7, we can define $k_n = \max\{k \geq 0 : \lambda_{\mathfrak{b},n}(k) = k^2\}$. By continuity, we have, for all $k > k_n$, $\lambda_{\mathfrak{b},n}(k) < k^2$. Let us now prove the uniqueness. Let us consider a solution $\tilde{k}_n \geq 0$. For all integer $p \geq p_0$ with p_0 large enough, we have $\lambda_{\mathfrak{b},n}(\tilde{k}_n + \frac{1}{p}) < (\tilde{k}_n + \frac{1}{p})^2$. Let us now consider the eigenvalue equation

$$D_x^2\varphi_{n,p} + (\tilde{k}_{n,p} - a_{\mathfrak{b}}(x))^2\varphi_{n,p} = \lambda_{n,p}\varphi_{n,p}, \quad (2.7)$$

where $\varphi_{n,p} = \psi_{\mathfrak{b},n}(\tilde{k}_{n,p})$, $\tilde{k}_{n,p} = \tilde{k}_n + \frac{1}{p}$ and $\lambda_{n,p} = \lambda_{\mathfrak{b},n}(\tilde{k}_{n,p})$. Let us investigate the limit $p \rightarrow +\infty$. As seen in the proof of Lemma 2.4, we know that there exists $\alpha \in \mathbb{R}^*$ such that, for $x \leq 0$,

$$\varphi_{n,p}(x) = \alpha e^{x\sqrt{\tilde{k}_{n,p}^2 - \lambda_{n,p}}}.$$

By solving (2.7) on $x \geq 0$ and using the parabolic cylinder function U , we find that there exists β such that, for $x \geq 0$,

$$\varphi_{n,p}(x) = \beta U\left(-\frac{\lambda_{n,p}}{2}; \sqrt{2}(x - \tilde{k}_{n,p})\right).$$

Since $\varphi_{n,p} \neq 0$, we have $(\alpha, \beta) \neq (0, 0)$ and $\varphi_{n,p}$ is C^1 at $x = 0$, we get the transmission condition:

$$\sqrt{\tilde{k}_{n,p}^2 - \lambda_{n,p}} \hat{U}\left(-\frac{\lambda_{n,p}}{2}; -\tilde{k}_{n,p}\sqrt{2}\right) - \sqrt{2}U'\left(-\frac{\lambda_{n,p}}{2}; -\tilde{k}_{n,p}\sqrt{2}\right) = 0.$$

By continuity and taking the limit $p \rightarrow +\infty$, we get

$$U'\left(-\frac{\tilde{k}_n^2}{2}; -\tilde{k}_n\sqrt{2}\right) = 0.$$

Notice that the function $x \mapsto U\left(-\frac{\tilde{k}_n^2}{2}; \sqrt{2}(x - \tilde{k}_n)\right)$ solves the differential equation

$$\begin{cases} -y''(x) + (x - \tilde{k}_n)^2 y(x) = \tilde{k}_n^2 y(x) \\ y'(0) = 0 \text{ and } y(0) \neq 0 \end{cases}$$

Moreover it belongs to $\mathcal{S}(\overline{\mathbb{R}_+})$. Therefore, there exists $\ell \geq 1$ such that $\mu_\ell^N(\tilde{k}_n) = \tilde{k}_n^2$, and therefore $\tilde{k}_n = \xi_{\ell-1}$ and $\tilde{k}_n^2 = \Theta_{\ell-1}$. By Proposition 2.8, we know that $\tilde{k}_n^2 = \lambda_{\mathfrak{b},n}(\tilde{k}_n) \in [E_{n-1}, E_n]$. Moreover, we recall that $\Theta_{\ell-1} \in (E_{\ell-1}, E_\ell)$. This implies that $\ell = n$. \square

3. TRAPPING MAGNETIC STEP

In this section, we focus on the case $\mathfrak{b} = (b_1, 1)$ with $-1 < b_1 < 0$. The case of $b_1 = -1$ is treated in [9]. In that case, due to the symmetry, all levels are split as $k \rightarrow +\infty$. The asymptotic expansion comes from [16, 25]. The band functions for this case are shown in Figure 5.

When $b_1 < 0$, the effective potential $V_{\mathfrak{b}}(x; k)$, described in (1.2), diverges to infinity as $x \rightarrow \pm\infty$, so that for all $k \in \mathbb{R}$, the operator $\mathfrak{h}_{\mathfrak{b}}(k)$ has compact resolvent. The spectrum of $\mathfrak{h}_{\mathfrak{b}}(k)$ is discrete and the eigenvalues $(\lambda_{\mathfrak{b},n}(k))_{n \geq 1}$ are all simple.

For $k < 0$, the effective potential $V_{\mathfrak{b}}(x; k)$ has the form of an asymmetric potential well, see Figure 2. It follows that $V_{\mathfrak{b}}(\cdot; k) \geq k^2$ so that, for $k \leq 0$, $\lambda_{\mathfrak{b},n}(k) \geq k^2$. In particular, we have, for all $n \geq 1$,

$$\lim_{k \rightarrow -\infty} \lambda_{\mathfrak{b},n}(k) = +\infty, \quad k < 0.$$

For $k > 0$, the effective potential has the form of an asymmetric double-well:

$$V_{\mathfrak{b}}(x; k) = \begin{cases} (k - b_1 x)^2 & x < 0 \\ (k - x)^2 & x > 0 \end{cases} \quad (3.8)$$

This asymmetric double-well admits two nondegenerate minima at $x = \frac{k}{b_1}$ and $x = k$, for $k > 0$ and $b_1 < 0$.

Let us now investigate the behavior of the band functions in the limit $k \rightarrow +\infty$ and prove Theorem 1.6. Due to the asymmetry of the double-well potential (3.8), we will prove, among other results, that not all the Landau levels exhibit an exponentially small splitting as in the case of $(b_1, b_2) = (-1, 1)$.

3.1. Limits of the band functions. Let us first describe the set of the limits of the band functions $\lambda_{\mathbf{b},n}(k)$ as $k \rightarrow +\infty$. As noted above, the limit as $k \rightarrow -\infty$ of any band function is plus infinity.

Lemma 3.1. *Assume that $\mathbf{b} = (b_1, 1)$, with $-1 < b_1 < 0$. Recall from Definition 2.7 that \mathfrak{L} denotes the union of the Landau levels for magnetic fields $b_1 \in (-1, 0)$ and $b_2 = 1$, and that \mathfrak{S} denotes the splitting set $\{E_N, n \geq 1\} \cap \{|b_1|E_n, n \geq 1\}$. Then the set of limits as $k \rightarrow +\infty$ of the band functions is precisely the set of Landau levels \mathcal{L} :*

$$\left\{ \lim_{k \rightarrow +\infty} \lambda_{\mathbf{b},n}(k), n \geq 1 \right\} = \mathcal{L}.$$

The band functions exhibit the following behavior as $k \rightarrow +\infty$:

- (1) *Non-splitting case.* If a band function limit $\lambda \in \mathfrak{L} \setminus \mathfrak{S}$, then for any $\epsilon_0 > 0$, there exists a $K > 0$ such that if $k \geq K$ then it holds that $\text{rank } \mathbb{1}_{[\lambda-\epsilon_0, \lambda+\epsilon_0]}(\mathfrak{h}_{\mathbf{b}}(k)) = 1$ and there exists a unique $n \geq 1$ such that $\lambda_{\mathbf{b},n}$ converges towards λ .
- (2) *Splitting case.* If a band function limit $\lambda \in \mathfrak{S}$, then for any $\epsilon_0 > 0$, there exists a $K > 0$ such that if $k \geq K$ then it holds that $\text{rank } \mathbb{1}_{[\lambda-\epsilon_0, \lambda+\epsilon_0]}(\mathfrak{h}_{\mathbf{b}}(k)) = 2$ and there exists a unique $n \geq 1$ such that $\lambda_{\mathbf{b},n}$ and $\lambda_{\mathbf{b},n+1}$ converge towards λ .

Proof. Let us first provide a semiclassical reformulation and summarize the standard procedure of the harmonic approximation for multiple wells (see [12, Chapter 4] for example). For $k > 0$, we let $x = ky$ and the operator $\mathfrak{h}_{\mathbf{b}}(k)$ is unitarily equivalent to $k^2 \tilde{\mathfrak{h}}_{\mathbf{b},k^{-2}}$, where $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar}$ is the self-adjoint realization on $L^2(\mathbb{R})$ of the differential operator $\hbar^2 D_y^2 + V_{\mathbf{b}}(y; 1)$. The minima of $V_{\mathbf{b}}(\cdot; 1)$ are reached for $x = b_1^{-1} < 0$ and $x = 1$. We have $V_{\mathbf{b}}''(1; 1) = 2$ and $V_{\mathbf{b}}''(b_1^{-1}; 1) = 2|b_1|$. Let us just recall the spirit of the analysis. Let us consider $\eta > 0$ such that $b_1^{-1} + \eta < 0$ and $1 - \eta > 0$. Then, let us consider the operator $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta} = \tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}^{\text{le}} \oplus \tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}^{\text{ri}}$ where $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar}^{\text{le}}$ and $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar}^{\text{ri}}$ are the Dirichlet realizations of $\hbar^2 D_y^2 + V(y; 1)$ on $L^2((-\infty, b_1 + \eta))$ and on $L^2((1 - \eta, +\infty))$ respectively. Let us now consider an arbitrary constant $C > 0$. It is well-known (see Helffer's lecture notes [12]) that any eigenfunction of $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}$ or of $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar}$ associated with an eigenvalue $\lambda \leq C\hbar$ satisfies decay estimates of Agmon type away from the minima. In particular, this allows to prove the existence of $c > 0$ such that $\text{sp}(\tilde{\mathfrak{h}}_{\mathbf{b},\hbar}) \cap \{\lambda \in \mathbb{R} : \lambda \leq C\hbar\}$ and $\text{sp}(\tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}) \cap \{\lambda \in \mathbb{R} : \lambda \leq C\hbar\}$ coincides modulo $\mathcal{O}(e^{-c\hbar^{-1}})$. Then, by the harmonic approximation, the spectrum of $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}$ is well-known in the semiclassical limit: the n -th eigenvalue of $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}^{\text{le}}$ is equivalent to $|b_1|E_n\hbar$ and the one of $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}^{\text{ri}}$ to $E_n\hbar$. The desired conclusion follows. The last part of the statement follows from the fact that the spectra (with multiplicity) of $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar}$ and $\tilde{\mathfrak{h}}_{\mathbf{b},\hbar,\eta}$ coincide modulo an exponentially small error. \square

As described in [16], the convergence of the band functions toward these limits corresponds to the existence of bulk states with energies close to the Landau levels. We next provide a more accurate description of the convergence of the band functions towards their limits. This is important for two reasons. First, this allows a more quantitative characterization of the associated bulk states following the analysis of [16]. Second, the manner in which the band function approaches its limit as $k \rightarrow +\infty$, that is, whether it approaches from above or below the Landau level, provides a hint as to whether the band function is globally monotonic or not. In

order to describe the asymptotic behavior of the band function, we will leave the semiclassical techniques and use the characterization of the eigenvalues $\lambda_{\mathfrak{b},n}(k)$ as zeros of certain special functions.

3.2. Splitting and non-splitting of levels. We consider $k > 0$ and calculate the behavior of the energy levels in the splitting and the non-splitting cases. This is a refinement of Lemma 3.1, where we have shown that each band function $\lambda_{\mathfrak{b},n}$ tends to a threshold that is an element of the set of Landau levels \mathfrak{L} . There is no explicit formula for the index of the corresponding Landau level as a function of n , the band index.

In this section, for the sake of simplicity, we will consider a converse strategy: Given a threshold E_n or bE_n , where $n \in \mathbb{N}^*$ is fixed, we describe the asymptotic behavior of the band functions converging toward this threshold. As described in Lemma 3.1, there are either one or two band functions $\lambda_{\mathfrak{b},p_n}(k)$ converging toward a threshold E_n or bE_n in \mathfrak{L} . Although the indices p_n of these band functions are not explicit in terms of the index n , the asymptotics of the related band functions $\lambda_{\mathfrak{b},p_n}(k)$, as $k \rightarrow +\infty$, are explicit functions of n .

Let us notice that this procedure will provide the asymptotics of all the band functions since there is a bijection between their limits and the set of the thresholds \mathfrak{L} (counted with multiplicity).

3.2.1. Algebraic characterization of the eigenvalues of $\mathfrak{h}_{\mathfrak{b}}(k)$. We begin with an algebraic characterization of the eigenvalues of $\mathfrak{h}_{\mathfrak{b}}(k)$. Recall that $U(a, x)$ is the first Weber parabolic cylindrical function introduced in Section 2 as a solution of (2.6).

Proposition 3.2. *Assume that $\mathfrak{b} = (b_1, 1)$, with $-1 < b_1 < 0$. A number λ is an eigenvalue of $\mathfrak{h}_{\mathfrak{b}}(k)$ for $k \in \mathbb{R}$ if and only if it is a solution of $D_{\mathfrak{b}}(\lambda, k) = 0$, where*

$$D_{\mathfrak{b}}(\lambda, k) := U\left(-\frac{\lambda}{2b}, -\sqrt{2}\frac{k}{\sqrt{b}}\right)U'\left(-\frac{\lambda}{2}, -\sqrt{2}k\right) \\ + \sqrt{b}U'\left(-\frac{\lambda}{2b}, -\sqrt{2}\frac{k}{\sqrt{b}}\right)U\left(-\frac{\lambda}{2}, -\sqrt{2}k\right)$$

and $b = |b_1|$.

Proof. From the eigenvalue equation and the form of the potential (3.8), we are led to solve for $x > 0$:

$$-y''(x) + (x - k)^2 y(x) = \lambda y(x)$$

with the condition that the solution is in $\mathbb{L}^2(\mathbb{R}_+)$. The only such solution is

$$x \mapsto U\left(-\frac{\lambda}{2}, \sqrt{2}(x - k)\right).$$

In particular, when $\lambda = 2n - 1$ is a Landau level, the solution is the n -th Hermite function centered at $x = k$. On the half line $x < 0$, we solve

$$-y''(x) + (b_1 x - k)^2 y(x) = \lambda y(x)$$

with the condition that the solution is in $\mathbb{L}^2(\mathbb{R}_-)$. This solution may be expressed as

$$x \mapsto U\left(-\frac{\lambda}{2b}, \sqrt{2}\left(-\sqrt{b}x - \frac{k}{\sqrt{b}}\right)\right).$$

Therefore, if $\psi_{\mathbf{b},n}(\cdot, k)$ is an eigenfunction associated with $\lambda = \lambda_{\mathbf{b},n}(k)$, it may be written as (up to an overall multiplicative constant):

$$\psi_{\mathbf{b},n}(x, k) = \begin{cases} U(-\frac{\lambda}{2b}, \sqrt{2}(-\sqrt{b}x - \frac{k}{\sqrt{b}})) & \text{if } x < 0 \\ \alpha U(-\frac{\lambda}{2}, \sqrt{2}(x - k)) & \text{if } x > 0 \end{cases}$$

where $\alpha \in \mathbb{R}$ is a constant. Since the eigenfunction is \mathcal{C}^1 at 0, we have

$$\begin{cases} U(-\frac{\lambda}{2b}, -\sqrt{2}\frac{k}{\sqrt{b}}) = \alpha U(-\frac{\lambda}{2}, -\sqrt{2}k) \\ -\sqrt{2b}U'(-\frac{\lambda}{2b}, -\sqrt{2}\frac{k}{\sqrt{b}}) = \alpha\sqrt{2}U'(-\frac{\lambda}{2}, -\sqrt{2}k) \end{cases}$$

It is possible to solve the above conditions if and only if the 2×2 matrix

$$\begin{pmatrix} U(-\frac{\lambda}{2b}, -\sqrt{2}\frac{k}{\sqrt{b}}) & U(-\frac{\lambda}{2}, -\sqrt{2}k) \\ -\sqrt{2b}U'(-\frac{\lambda}{2b}, -\sqrt{2}\frac{k}{\sqrt{b}}) & U'(-\frac{\lambda}{2}, -\sqrt{2}k) \end{pmatrix}$$

is non-invertible. \square

We will use the following result on the asymptotic expansion of the determinant $D_{\mathbf{b}}(\lambda, k)$ as $k \rightarrow +\infty$.

Lemma 3.3. *Assume that $\mathbf{b} = (b_1, 1)$, with $-1 < b_1 < 0$, and let $b = |b_1|$. The determinant $D_{\mathbf{b}}(\lambda, k)$ of Proposition 3.2 can be written as*

$$\begin{aligned} D_{\mathbf{b}}(\lambda, k) &= e^{(1+\frac{1}{b})\frac{k^2}{2}} T_1(\lambda, k) + e^{(-1+\frac{1}{b})\frac{k^2}{2}} T_2(\lambda, k) \\ &\quad + e^{(1-\frac{1}{b})\frac{k^2}{2}} T_3(\lambda, k) + e^{(-1-\frac{1}{b})\frac{k^2}{2}} T_4(\lambda, k), \end{aligned} \quad (3.9)$$

where the factors $T_j(\lambda, k)$ admit the following asymptotic expansions as $k \rightarrow +\infty$:

$$\begin{cases} T_1(\lambda, k) = \frac{1}{\Gamma(\frac{b-\lambda}{2b})\Gamma(\frac{1-\lambda}{2})} k^{-\frac{b+1}{2b}\lambda} 2^{\frac{3}{2} - \frac{(1+b)\lambda}{4b}} b^{\frac{b+\lambda}{4b}} \pi (1 + \mathcal{O}(k^{-2})), \\ T_2(\lambda, k) = \frac{1}{\Gamma(\frac{b-\lambda}{2b})} k^{\frac{b-1}{2b}\lambda-2} \left(\frac{\sqrt{\pi}}{2}\right) 2^{\lambda\frac{b-1}{4b}} b^{\frac{b+\lambda}{4b}} (1+b) \left(-\sin \frac{\lambda\pi}{2}\right) (1 + \mathcal{O}(k^{-2})), \\ T_3(\lambda, k) = \frac{1}{\Gamma(\frac{1-\lambda}{2})} k^{\frac{-b+1}{2b}\lambda-2} \left(\frac{\sqrt{\pi}}{2}\right) 2^{\lambda\frac{-b+1}{4b}} b^{\frac{b-\lambda}{4b}} (1+b) \left(-\sin \frac{\lambda\pi}{2b}\right) (1 + \mathcal{O}(k^{-2})), \\ T_4(\lambda, k) = \sqrt{2} k^{\frac{b+1}{2b}\lambda} 2^{\lambda\frac{b+1}{4b}} b^{\frac{b-\lambda}{4b}} \cos \frac{(1+b)\lambda\pi}{2b} (1 + \mathcal{O}(k^{-2})). \end{cases}$$

Proof. These expansions follow by the asymptotic expansion of the Weber function U . We refer the appendix, section A, and to [2, Chapter 19]. \square

Remark 3.4. We have controlled these rather tedious calculations with the computer software *Mathematica*.

Remark 3.5. To obtain the large k asymptotics of $D_{\mathbf{b}}(\lambda, k)$, we factor out the leading exponential factor $e^{(1+\frac{1}{b})\frac{k^2}{2}}$ from the right side of (3.9). The zeros of $D_{\mathbf{b}}(\lambda, k)$ are the same as the zeros of

$$\tilde{D}_{\mathbf{b}}(\lambda, k) := e^{-(1+\frac{1}{b})\frac{k^2}{2}} D_{\mathbf{b}}(\lambda, k),$$

where

$$\tilde{D}_{\mathbf{b}}(\lambda, k) = T_1(\lambda, k) + e^{-k^2} T_2(\lambda, k) + e^{-\frac{k^2}{b}} T_3(\lambda, k) + e^{-(1+\frac{1}{b})k^2} T_4(\lambda, k). \quad (3.10)$$

We next note that the functions T_1 and T_2 are bounded as $k \rightarrow +\infty$. The coefficient T_3 grows like $k^{(\frac{1-b}{2b})\lambda-2}$ for λ large enough. Similarly, the term T_4 is increasing in k . However, the prefactors of these terms exhibit Gaussian decay.

The strategy of the proof of Theorem 1.6 consists of writing the band function $\lambda_{b,p_n}(k)$ as

$$\lambda_{b,p_n}(k) = \tilde{E}_n + \varepsilon_n(k),$$

where $\lim_{k \rightarrow +\infty} \lambda_{b,p_n}(k) = \tilde{E}_n$, with either $\tilde{E}_n = E_n$ or $\tilde{E}_n = bE_n$, and $\lim_{k \rightarrow +\infty} \varepsilon_n(k) = 0$. For all $k \in \mathbb{R}$, we have

$$\tilde{D}_b(\lambda_{b,p_n}(k), k) = \tilde{D}_b(\tilde{E}_n + \varepsilon_n(k), k) = 0.$$

The distinction between the non-splitting and splitting cases is encoded in the gamma function coefficients of the $T_j(\lambda, k)$ terms in Lemma 3.3.

In the **non-splitting case**, we have for one of the gamma functions

$$\Gamma\left(\frac{b - \lambda_{b,p_n}(k)}{2b}\right) \rightarrow \Gamma\left(\frac{b - \tilde{E}_n}{2b}\right).$$

In case (ia), the argument of the gamma function is $((b+1)/2 - n)/b$ that is never a negative integer so Γ^{-1} does not have a zero and does not contribute to the asymptotics. In case (ib), the argument of this gamma functions is $(1-n)$ so there are poles of the gamma function. The corresponding zeros of Γ^{-1} contribute to the vanishing rate of T_1 and T_2 . As for the other gamma function, we have

$$\Gamma\left(\frac{1 - \lambda_{b,p_n}(k)}{2}\right) \rightarrow \begin{cases} \Gamma(1-n) & \text{(ia)} \\ \Gamma(\frac{b+1}{2} - nb) & \text{(ib)} \end{cases},$$

so there is a pole in case (ia). These asymptotics are developed in Section 3.2.2.

In the **splitting case**, we have that $b = E_n/E_m$, and it follows that both gamma functions have poles. Consequently, both factors Γ^{-1} have a zero as $k \rightarrow +\infty$ and these contribute to the asymptotics as developed in Section 3.2.3. This analysis also shows that $T_1(\tilde{E}_n, k) = 0$ since for (ia) the inverse of the gamma function $\Gamma(\frac{1-\lambda}{2})$ vanishes for $\lambda = E_n$ and in the case (ib) the inverse of the other gamma function $\Gamma(\frac{b-\lambda}{2b}) = 0$ for $\lambda = bE_n$.

3.2.2. No splitting of levels. Let us prove Item (i) in Theorem 1.6. The existence of p_n comes from Lemma 3.1. Let us assume case (ia), the proof of case (ib) being symmetric. Then

$$\lim_{k \rightarrow +\infty} \frac{1}{\Gamma(\frac{b - \lambda_{b,p_n}(k)}{2b})} = \frac{1}{\Gamma(\frac{b - E_n}{2b})} \neq 0, \quad (3.11)$$

the last relation coming from the facts that the only roots of $1/\Gamma$ are the negative integers and that $\frac{b - E_n}{2} \notin -\mathbb{N}$, since this is not a splitting case. Recall that the roots of $1/\Gamma$ are simple. In particular, as k gets large,

$$\begin{aligned} \frac{1}{\Gamma(\frac{1 - \lambda_{b,p_n}(k)}{2})} &= -\frac{\varepsilon_n(k)}{2} \left(\frac{1}{\Gamma}\right)'(1-n)(1 + O(\varepsilon_n(k))) \\ &= (-\frac{\varepsilon_n(k)}{2})(-1)^{n-1}(n-1)!(1 + o(1)). \end{aligned} \quad (3.12)$$

As mentioned above, we replace $\lambda_{b,p_n}(k)$ by $E_n + \varepsilon_n(k)$ in the equation $\tilde{D}_b(\lambda_{b,p_n}(k), k) = 0$ and obtain an equation in the form

$$\hat{\alpha}(k)\varepsilon_n(k) + \hat{\beta}(k) = 0,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are analytic functions such that

$$\hat{\alpha}(k) \underset{k \rightarrow +\infty}{\sim} -\frac{1}{2}(-1)^{n-1}(n-1)! \frac{1}{\Gamma(\frac{b-E_n}{2b})} k^{-\frac{b+1}{2b}E_n} 2^{\frac{3}{2}-\frac{(1+b)E_n}{4b}} b^{\frac{b+E_n}{4b}} \pi,$$

$$\hat{\beta}(k) \underset{k \rightarrow +\infty}{\sim} -\frac{1}{\Gamma(\frac{b-E_n}{2b})} k^{\frac{b-1}{2b}E_n-2} \left(\frac{\sqrt{\pi}}{2}\right) 2^{E_n \frac{b-1}{4b}} b^{\frac{b+E_n}{4b}} (1+b) \sin\left(\frac{E_n \pi}{2}\right) e^{-k^2},$$

obtained by using Lemma 3.3. We deduce the asymptotic behavior of $\varepsilon_n(k)$ as $k \rightarrow +\infty$.

Since $\varepsilon_n(k)$ is negative the corresponding band functions $\lambda_{b,n}(k)$ approach the Landau level E_n or bE_n as $k \rightarrow +\infty$ from below. Taking into account the fact that $\lambda_{b,p_n}(k) \rightarrow +\infty$ as $k \rightarrow -\infty$, this means that these band functions are not monotonic and have at least one minimum.

3.2.3. Splitting of levels. Let us finally give the proof of Item (ii) in Theorem 1.6. When the condition for splitting is achieved, two band functions have the same limit as $k \rightarrow +\infty$. We prove that one level approaches the common Landau level from below, and hence has a minimum, whereas the other band function tends to its limit by above.

The existence of p_n is a consequence of Lemma 3.1. We also know from this lemma that both functions ε_n^\pm tends to 0 as $k \rightarrow +\infty$. We use the generic notation $\lambda(k) = E_n + \varepsilon(k)$ satified by these two functions. Recalling that $b = \frac{E_n}{E_m}$, we note that as k tends to $+\infty$:

$$\frac{1}{\Gamma\left(\frac{b-\lambda(k)}{2b}\right)} = \frac{1}{\Gamma\left(1-m-\frac{\varepsilon(k)}{2b}\right)} = -\frac{\varepsilon(k)}{2b} \left(\frac{1}{\Gamma}\right)' (1-m) + \mathcal{O}(\varepsilon(k)^2)$$

and

$$\frac{1}{\Gamma\left(\frac{1-\lambda(k)}{2}\right)} = \frac{1}{\Gamma\left(1-n-\frac{\varepsilon(k)}{2}\right)} = -\frac{\varepsilon(k)}{2} \left(\frac{1}{\Gamma}\right)' (1-n) + \mathcal{O}(\varepsilon(k)^2).$$

Substituting these Taylor expansions into (3.9), the condition $D_b(\lambda(k), k) = 0$ implies that ε_n^\pm are the roots of the equation

$$\hat{A}\varepsilon^2 + \varepsilon(\hat{B} + \hat{C}) + \hat{D} = 0, \quad (3.13)$$

where $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ are analytic functions of k respectively equivalent to the following functions as $k \rightarrow +\infty$:

$$\begin{cases} A(k) = (-1)^{n+m}(n-1)!(m-1)!\pi 2^{-\frac{1}{2}-\frac{(1+b)E_n}{4b}} b^{\frac{m}{2}-1} k^{-\frac{b+1}{2b}E_n}, \\ B(k) = (-1)^{n+m}(m-1)!\sqrt{\pi} 2^{-3/2+\frac{(b-1)E_n}{4b}} b^{\frac{b+E_n}{4b}-1} (1+b) k^{\frac{b-1}{2b}E_n-2} e^{-k^2}, \\ C(k) = (-1)^{n+m}(n-1)!\sqrt{\pi} 2^{-3/2+\frac{(1-b)E_n}{4b}} b^{\frac{b-E_n}{4b}-1} (1+b) k^{\frac{-b+1}{2b}E_n-2} e^{-k^2/b}, \\ D(k) = (-1)^{n+m-1} 2^{\frac{E_n+b(2+E_n)}{4b}} b^{\frac{b-E_n}{4b}} k^{\frac{b+1}{2b}E_n} e^{-k^2(1+1/b)}. \end{cases} \quad (3.14)$$

The discriminant

$$\Delta = (\hat{B} + \hat{C})^2 - 4\hat{A}\hat{D}$$

is positive for k large enough since there hold $\hat{B}(k) \gg \hat{C}(k)$ and $\hat{B}(k)^2 \gg \hat{D}(k)$ as k goes to $+\infty$ (notice also that $\hat{A}(k) = o(1)$). In particular, the two roots of (3.13) admit the following expansions:

$$\varepsilon^+ = -\frac{\hat{D}}{\hat{B}}(1 + o(1)) \quad \text{and} \quad \varepsilon^- = -\frac{\hat{B}}{\hat{A}}(1 + o(1)).$$

Therefore, using the equivalents given in (3.14), we get as k gets large:

$$\begin{aligned} \varepsilon^+(k) &= \frac{b^{-m+\frac{3}{2}} 2^{m+\frac{3}{2}}}{(m-1)!(1+b)\sqrt{\pi}} k^{2m+1} e^{-k^2/b} (1 + o(1)), \\ \varepsilon^-(k) &= -2^{n-3/2} \frac{(1+b)}{(n-1)!\sqrt{\pi}} k^{2n-3} e^{-k^2} (1 + o(1)). \end{aligned}$$

By definition, $\varepsilon_n^- < \varepsilon_n^+$. Due to the obvious sign of ε^+ and ε^- for k large enough, we get $\varepsilon^\pm = \varepsilon_n^\pm$ and the result follows from the above asymptotic expansions.

APPENDIX A. PARABOLIC CYLINDER FUNCTIONS

We list some properties of parabolic cylinder functions, and refer to [2, Chapter 19]. Given $a \in \mathbb{R}$, denote by $U(a, x)$ and $V(a, x)$ the so-called standard solutions of the differential equation

$$-y''(x) + \frac{1}{4}x^2 y(x) = -ay(x), \quad x \in \mathbb{R}.$$

It follows by the series expansions of U and V that they are real-valued if $a \in \mathbb{R}$ and $x \in \mathbb{R}$. Moreover, as $x \rightarrow +\infty$,

$$U(a, x) \sim \exp(-x^2/4) x^{-a-1/2} \sum_{j=0}^{+\infty} (-1)^j \frac{(\frac{1}{2} + a)_{2j}}{j!(2x^2)^j} \quad (\text{A.15})$$

and

$$V(a, x) \sim \sqrt{\frac{2}{\pi}} \exp(x^2/4) x^{a-1/2} \sum_{j=0}^{+\infty} \frac{(\frac{1}{2} - a)_{2j}}{j!(2x^2)^j}. \quad (\text{A.16})$$

Here $(x)_j = x(x-1)(x-2)\cdots(x-j+1)$ denotes the Pochhammer symbol. It also holds that

$$\pi V(a, x) = \Gamma(a + \frac{1}{2}) \{ \sin(\pi a) U(a, x) + U(a, -x) \}. \quad (\text{A.17})$$

This enables us to give an asymptotic expansion of $U(a, x)$ as $x \rightarrow -\infty$. Indeed, as $x \rightarrow +\infty$,

$$\begin{aligned} U(a, -x) &\sim \frac{\sqrt{2\pi}}{\Gamma(a + \frac{1}{2})} \exp(x^2/4) x^{a-1/2} \sum_{j=0}^{+\infty} \frac{(\frac{1}{2} - a)_{2j}}{j!(2x^2)^j} \\ &\quad - \sin(\pi a) \exp(-x^2/4) x^{-a-1/2} \sum_{j=0}^{+\infty} (-1)^j \frac{(\frac{1}{2} + a)_{2j}}{j!(2x^2)^j}. \end{aligned} \quad (\text{A.18})$$

If we keep only two terms in the sum,

$$\begin{aligned} U(a, -x) &\sim \frac{\sqrt{2\pi}}{\Gamma(a + \frac{1}{2})} \exp(x^2/4) x^{a-1/2} \left(1 + \frac{1}{2} \left(\frac{1}{2} - a \right)_2 \frac{1}{x^2} + \mathcal{O}(1/x^4) \right) \\ &\quad - \sin(\pi a) \exp(-x^2/4) x^{-a-1/2} \left(1 - \frac{1}{2} \left(a + \frac{1}{2} \right)_2 \frac{1}{x^2} + \mathcal{O}(1/x^4) \right). \end{aligned} \quad (\text{A.19})$$

Next, it holds that

$$U'(a, x) := \frac{d}{dx}U(a, x) = \frac{x}{2}U(a, x) - U(a-1, x),$$

and thus

$$U'(a, -x) = -\frac{x}{2}U(a, -x) - U(a-1, -x).$$

Combining this with (A.18), we have, as $x \rightarrow +\infty$,

$$\begin{aligned} U'(a, -x) &\sim -\frac{\sqrt{2\pi}}{\Gamma(a + \frac{1}{2})} \exp(x^2/4) x^{a+1/2} \times \\ &\quad \left[\frac{1}{2} \sum_{j=0}^{+\infty} \frac{(\frac{1}{2} - a)_{2j}}{j!(2x^2)^j} + \frac{a - \frac{1}{2}}{x^2} \sum_{j=0}^{+\infty} \frac{(\frac{3}{2} - a)_{2j}}{j!(2x^2)^j} \right] \\ &\quad + \sin(\pi a) \exp(-x^2/4) x^{-a+1/2} \times \\ &\quad \left[\frac{1}{2} \sum_{j=0}^{+\infty} (-1)^j \frac{(a + \frac{1}{2})_{2j}}{j!(2x^2)^j} - \sum_{j=0}^{+\infty} (-1)^j \frac{(a - \frac{1}{2})_{2j}}{j!(2x^2)^j} \right]. \end{aligned} \quad (\text{A.20})$$

With two terms in the sum only,

$$\begin{aligned} U'(a, -x) &\sim -\frac{\sqrt{2\pi}}{\Gamma(a + \frac{1}{2})} \exp(x^2/4) x^{a+1/2} \times \\ &\quad \left(\frac{1}{2} + \left\{ \frac{1}{4} \left(\frac{1}{2} - a \right)_2 + \left(a - \frac{1}{2} \right) \right\} \frac{1}{x^2} + \mathcal{O}(1/x^4) \right) \\ &\quad + \sin(\pi a) \exp(-x^2/4) x^{-a+1/2} \times \\ &\quad \left(-\frac{1}{2} + \left\{ -\frac{1}{4} \left(a + \frac{1}{2} \right)_2 + \frac{1}{2} \left(a - \frac{1}{2} \right)_2 \right\} \frac{1}{x^2} + \mathcal{O}(1/x^4) \right). \end{aligned} \quad (\text{A.21})$$

REFERENCES

- [1] *Digital library of mathematical functions*. National Institute of Standards and Technology. <http://dlmf.nist.gov/>.
- [2] M. ABRAMOWITZ, I. STEGUN. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964; available free online: <http://people.math.sfu.ca/~cbm/aands/>.
- [3] V. BONNAILLIE-NOËL, M. DAUGE, N. POPOFF. Ground state energy of the magnetic laplacian on general three-dimensional corner domains. *arXiv:1403.7043* (2014).
- [4] V. BONNAILLIE-NOËL, F. HÉRAU, N. RAYMOND. Magnetic WKB constructions. *arXiv:1405.7157* (2014).
- [5] M. DAUGE, B. HELFFER. Eigenvalues variation. I. Neumann problem for Sturm-Liouville operators. *J. Differential Equations* **104**(2) (1993) 243–262.
- [6] S. DE BIEVRE, J. PULE. Propagating edge states for a magnetic hamiltonian. *Math. Phys. Electron. J.* **5**(3) (1999).
- [7] M. DIMASSI, J. SJÖSTRAND. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge 1999.
- [8] N. DOMBROWSKI, F. GERMINET, G. RAIKOV. Quantization of edge currents along magnetic barriers and magnetic guides. *Ann. Henri Poincaré* **12**(6) (2011) 1169–1197.
- [9] N. DOMBROWSKI, P. D. HISLOP, E. SOCCORSI. Edge currents and eigenvalue estimates for magnetic barrier Schrödinger operators. *Asymptotic Analysis* **89**(3-4) (2014) 331–363.
- [10] S. FOURNAIS, B. HELFFER. *Spectral methods in surface superconductivity*. Progress in Non-linear Differential Equations and their Applications, 77. Birkhäuser Boston Inc., Boston, MA 2010.

- [11] T. GHOSH, A. DE MARTINO, W. HÄUSLER, L. DELL'ANNA, R. EGGER. Conductance quantization and snake states in graphene magnetic waveguides. *Physical Review B* **77**(8) (2008) 081404.
- [12] B. HELFFER. *Semi-classical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin 1988.
- [13] B. HELFFER, A. MORAME. Magnetic bottles in connection with superconductivity. *J. Funct. Anal.* **185**(2) (2001) 604–680.
- [14] B. HELFFER, J. SJÖSTRAND. Multiple wells in the semiclassical limit. I. *Comm. Partial Differential Equations* **9**(4) (1984) 337–408.
- [15] B. HELFFER, J. SJÖSTRAND. Multiple wells in the semi-classical limit III-interaction through non-resonant wells. *Mathematische Nachrichten* **124**(1) (1985) 263–313.
- [16] P. D. HISLOP, N. POPOFF, E. SOCCORSI. Characterization of bulk states in one-edge quantum hall systems. *Annales de l'IHP* (2014).
- [17] P. D. HISLOP, E. SOCCORSI. Edge states induced by Iwatsuka hamiltonians with positive magnetic fields. *Journal of Mathematical Analysis and Applications* **422**(1) (2015) 594–624.
- [18] P. D. HISLOP, E. SOCCORSI. Edge states induced by Iwatsuka Hamiltonians with positive magnetic fields. *J. Math. Anal. Appl.* (to appear 2014).
- [19] V. IVRII. *Microlocal Analysis, Sharp Spectral Asymptotics and Applications*. Available online on the author's website.
- [20] A. IWATSUKA. Examples of absolutely continuous Schrödinger operators in magnetic fields. *Publ. Res. Inst. Math. Sci.* **21**(2) (1985) 385–401.
- [21] M. MANTOU, R. PURICE. Some propagation properties of the Iwatsuka model. *Communications in mathematical physics* **188**(3) (1997) 691–708.
- [22] L. OROSZLÁNY, P. RAKYTA, A. KORMÁNYOS, C. J. LAMBERT, J. CSERTI. Theory of snake states in graphene. *Physical Review B* **77**(8) (2008) 081403.
- [23] F. PEETERS, A. MATULIS. Quantum structures created by nonhomogeneous magnetic fields. *Phys. Rev. B* **48** (1993) 15166–15174.
- [24] F. PEETERS, P. VASILOPOULOS. Quantum transport of a two-dimensional electron gas in a spatially modulated magnetic field. *Phys. Rev. B* **47** (1993) 1466–1473.
- [25] N. POPOFF. *Sur l'opérateur de Schrödinger magnétique dans un domaine diédral*. PhD thesis, Université de Rennes 1 2012.
- [26] N. POPOFF, E. SOCCORSI. Limiting absorption principle for the magnetic dirichlet laplacian in a half-plane. *arXiv:1409.6082* (2014).
- [27] G. RAIKOV. Eigenvalue asymptotics for the Schrödinger operator with perturbed periodic potential. *Invent. Math.* **110**(1) (1992) 75–93.
- [28] N. RAYMOND. *Little Magnetic Book*. arXiv:1405.7912 2014.
- [29] M. REED, B. SIMON. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York 1978.
- [30] J. RELNIERS, F. PEETERS. Snake orbits and related magnetic edge states. *Journal of Physics: Condensed Matter* **12**(47) (2000) 9771.
- [31] D. SAINT-JAMES, G. SARMA, E. THOMAS. *Type II Superconductivity*. Pergamon, Oxford 1969.
- [32] B. SIMON. Sturm oscillation and comparison theorems. In *Sturm-Liouville theory*, pages 29–43. Birkhäuser, Basel 2005.
- [33] E. SOCCORSI. Analyticity and asymptotic properties of the Maxwell operator's dispersion curves. *Mathematical Models and Methods in Applied Sciences* **11**(03) (2001) 387–406.
- [34] D. YAFAEV. On spectral properties of translationally invariant magnetic schrödinger operators. *Annales Henri Poincaré* **9**(1) (2008) 181–207.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506-0027, USA

E-mail address: `peter.hislop@uky.edu`

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE BORDEAUX

E-mail address: `nicolas.popoff@cpt.univ-mrs.fr`

INSTITUT DE RECHERCHE MATHÉMATIQUE DE RENNES. UNIVERSITÉ DE RENNES 1, FRANCE

E-mail address: `nicolas.raymond@univ-rennes1.fr`

CENTRE FOR MATHEMATICAL SCIENCES, BOX 118, SE-22100, LUND, SWEDEN

E-mail address: `mickep@maths.lth.se`